

A nonlinear Hamiltonian formalism for singular Lagrangian theories

Steven Duplij

Theory Group, Nuclear Physics Laboratory,
V. N. Karazin Kharkov National University
Svoboda Sq. 4, Kharkov 61077, Ukraine

steven.a.duplij@univer.kharkov.ua, sduplij@gmail.com
<http://webusers.physics.umn.edu/~duplij>

September 15, 2009

Abstract

We introduce a “nonlinear” version of the Hamiltonian formalism which allows a self-consistent description of theories with degenerate Lagrangian. A generalization of the Legendre transform to the case when the Hessian is zero is done using the mixed (envelope/general) solutions of the multidimensional Clairaut equation. The corresponding system of equations of motion is equivalent to the Lagrange equations, but contains “nondynamical” momenta and unresolved velocities. This system is reduced to the physical phase space and presented in the Hamiltonian form by introducing a new (non-Lie) bracket.

Our previous result [1] was in generalizing the Legendre transformation to singular (degenerate) Lagrangians (with zero Hessian matrix). For that a mixed (general/envelope) solution of the multidimensional Clairaut equation was introduced.

In this paper we apply the above idea to construct a self-consistent “nonlinear” version of the canonical (Hamiltonian) formalism and present an algorithm to describe any singular Lagrangian system without introducing constraints [2]. To simplify matters we use coordinates, but all the statements can be readily converted to coordinate free setting [3, 4]. We also consider systems with finite number of degrees of freedom. This is sufficient to explore the main ideas and constructions (this can be rendered to a field theory, e.g., using De Witt’s notation [5]).

First, recall the standard Legendre(-Fenchel) transform for the theory with nonsingular Lagrangian [6]. We then show its relation to the Clairaut equation [7] in some details [1], which will be used to explain the main idea below. Let¹ $L(q^A, v^A)$, $A = 1, \dots, n$, be a Lagrangian given by a smooth function² of $2n$ variables (n generalized coordinates q^A and n velocities $v^A = \dot{q}^A = dq^A/dt$) on the configuration space TM . By definition, a Hamiltonian $H(q^A, p_A)$ as a dual function to the Lagrangian (in the second set of variables p_A) constructed by means of the Legendre(-Fenchel) transform has the form³

$$H(q^A, p_A) = \sup_{v^A} [p_B v^B - L(q^A, v^A)], \quad (1)$$

where the supremum is taken with fixed q^A and p_A . In doing the Legendre transform, the coordinates q^A are treated as fixed (passive) parameters of the duality transformation, and velocities, v^A are *independent* functions of time. Then (1) leads to the *supremum condition*

$$p_A = \frac{\partial L(q^A, v^A)}{\partial v^A}. \quad (2)$$

To obtain a dual function $H(q^A, p_A)$, we need to get rid of dependence on velocity in the r.h.s. of (1). This can be done in *two* ways:

1) **Direct** way: resolve the condition (2) directly and obtain its solution as a set of functions $v^A = V^A(q^A, p_A)$, then substitute them to (1) and obtain the standard Hamiltonian on the physical phase space T^*M (see e.g. [6])

$$H^{st}(q^A, p_A) = p_B V^B(q^A, p_A) - L(q^A, V^A(q^A, p_A)). \quad (3)$$

This can be done only in the case of convex Lagrangian function (in the second set of variables v^A), which is equivalent to the Hessian being non-zero

$$\det \left\| \frac{\partial^2 L(q^A, v^A)}{\partial v^A \partial v^B} \right\| \neq 0. \quad (4)$$

¹We use indices in arguments, because by type of index we will distinguish them below.

²We consider time-independent case for simplicity and conciseness.

³We use summation convention for indices which are not in arguments of functions.

2) **Indirect** way: differentiate both sides of (4) by momenta and use the supremum condition (2) to obtain the “dual supremum condition” in the form

$$v^A = V^A(q^A, p_A) = \frac{\partial H(q^A, p_A)}{\partial p_A}. \quad (5)$$

Then we substitute these velocities to (4), which results in *no manifest dependence* of v^A . Thus we obtain a partial differential equation with respect to Hamiltonian which in fact is the multidimensional Clairaut equation

$$H^{cl}(q^A, \bar{p}_A) = \bar{p}_B \frac{\partial H^{cl}(q^A, \bar{p}_A)}{\partial \bar{p}_B} - L\left(q^A, \frac{\partial H^{cl}(q^A, \bar{p}_A)}{\partial \bar{p}_A}\right). \quad (6)$$

We call the transformation defined by (6) a *Clairaut duality transform* (or the *Clairaut-Legendre transform*) and $H^{cl}(q^A, \bar{p}_A)$ a *Clairaut-Hamilton function*. Note that (2) is normally treated as a *definition* of dynamical momenta p_A , but we should distinguish them from the parameters of the Clairaut duality transform \bar{p}_A : *before* applying the supremum condition (2) they are assumed *noncoincidental*.

The difference between the above two approaches is crucial for singular Lagrangian theories. We thus label the resulting Hamiltonians by different indices. Specifically, the Clairaut equation (6) *has solutions* even in the case when the Hessian (4) is zero. So the Clairaut duality transform is more general and includes the ordinary duality (Legendre-Fenchel) transform as a particular case. To show this and find solutions of the Clairaut equation (6), we differentiate it by \bar{p}_A to obtain

$$\left[\bar{p}_B - \frac{\partial L(q^A, v^A)}{\partial v^B} \right] \bigg|_{v^A = \frac{\partial H^{cl}(q^A, \bar{p}_A)}{\partial \bar{p}_A}} \cdot \frac{\partial^2 H^{cl}(q^A, \bar{p}_A)}{\partial \bar{p}_A \partial \bar{p}_B} = 0. \quad (7)$$

So we have two possibilities depending on which multiplier in (7) is zero:

1) Envelope solutions defined by the first multiplier in (7) being zero, this demand coincides with the supremum condition (2). So we obtain the standard Hamiltonian (3)

$$H_{env}^{cl}(q^A, \bar{p}_A) |_{\bar{p}_A = p_A} = H^{st}(q^A, p_A). \quad (8)$$

2) A general solution defined the “dual Hessian” being zero

$$\frac{\partial^2 H^{cl}(q^A, \bar{p}_A)}{\partial \bar{p}_A \partial \bar{p}_B} = 0. \quad (9)$$

This gives $\frac{\partial H^{cl}(q^A, \bar{p}_A)}{\partial \bar{p}_A} = c^A$ and then the general solution acquires the form

$$H_{gen}^{cl}(q^A, \bar{p}_A) = \bar{p}_B c^B - L(q^A, c^A), \quad (10)$$

where c^A are arbitrary smooth functions considered in the Clairaut equation (6) as parameters. Note that $H_{gen}^{cl}(q^A, \bar{p}_A)$ is always linear in the variables \bar{p}_A

which now do not coincide with the dynamical momenta, because we do not have the supremum condition (2).

Now consider a singular Lagrangian theory for which the Hessian (4) is zero. This means that the rank of Hessian matrix $W_{AB} = \frac{\partial^2 L(q^A, v^A)}{\partial v^A \partial v^B}$ is $r < n$, and we suppose that r is constant. We rearrange indices of W_{AB} in such a way that a nonsingular minor of rank r appears in the upper left corner. Represent the index A as follows: if $A = 1, \dots, r$, we replace A with i (the “regular” index), and, if $A = r + 1, \dots, n$ we replace A with α (the “degenerate” index). Obviously, $\det W_{ij} \neq 0$, and $\text{rank } W_{ij} = r$. Thus any set of variables labelled by a single index splits as a disjoint union of two subsets. We call those subsets *regular* (having Latin indices) and *degenerate* (having Greek indices).

The standard Legendre transform is not applicable in the singular case because the condition (4) is not valid [4]. Therefore the supremum condition (2) cannot be resolved under degenerate A , but it can be resolved under regular A only, because $\det W_{ij} \neq 0$. On the contrary, the Clairaut duality transform given by (6) independent of the Hessian being zero or not [1]. Thus, we state the main assumption of the formalism we present here: *the ordinary duality of convex functions can be generalized to the Clairaut duality for functions with zero Hessian*. This can be rephrased by saying that the standard Legendre(-Fenchel) transform of nonsingular Lagrangian theory is generalized to the Clairaut-Legendre transform, and in both cases the corresponding transformation is described by the same Clairaut equation (6).

To find its solutions, we again differentiate (6) by \bar{p}_A and present the sum (7) in B in two terms: regular and degenerate ones

$$\left[\bar{p}_i - \frac{\partial L(q^A, v^A)}{\partial v^i} \right] \cdot \frac{\partial^2 H^{cl}(q^A, \bar{p}_A)}{\partial \bar{p}_A \partial \bar{p}_i} + \left[\bar{p}_\alpha - \frac{\partial L(q^A, v^A)}{\partial v^\alpha} \right] \cdot \frac{\partial^2 H^{cl}(q^A, \bar{p}_A)}{\partial \bar{p}_A \partial \bar{p}_\alpha} = 0. \quad (11)$$

As $\det W_{ij} \neq 0$, we suggest to replace (11) by the conditions

$$\bar{p}_i = p_i = \frac{\partial L(q^A, v^A)}{\partial v^i}, \quad (12)$$

$$\frac{\partial^2 H^{cl}(q^A, \bar{p}_A)}{\partial \bar{p}_A \partial \bar{p}_\alpha} = 0. \quad (13)$$

In this way we obtain a “mixed” envelope/general solution of the Clairaut equation, which can be also treated as a “partial” Legendre transform [1].

After resolving of (12) under regular velocities $v^i = V^i(q^A, p_i, v^\alpha)$ and writing down a solution of (13) as $\frac{\partial H^{cl}(q^A, \bar{p}_A)}{\partial \bar{p}_\alpha} = v^\alpha$ (where v^α are arbitrary functions, unresolved velocities) we obtain a “mixed” Clairaut-Hamilton function

$$H_{mix}^{cl}(q^A, p_i, \bar{p}_\alpha, v^\alpha) = p_i V^i(q^A, p_i, v^\alpha) + \bar{p}_\alpha v^\alpha - L(q^A, V^i(q^A, p_i, v^\alpha), v^\alpha), \quad (14)$$

which is the desired Clairaut-Legendre transform written in coordinates. Note that (14) coincides with the “slow and careful Legendre transformation” of [8] and with the “generalized Legendre transformation” of [9].

The standard Lagrange equations of motion $\frac{d}{dt} \frac{\partial L(q^A, v^A)}{\partial v^A} = \frac{\partial L(q^A, v^A)}{\partial q^A}$ in our notation have the form

$$\frac{dp_i}{dt} = \frac{\partial L(q^A, v^A)}{\partial q^i}, \quad \frac{dh_\alpha(q^A, p_i)}{dt} = - \frac{\partial L(q^A, v^A)}{\partial q^\alpha} \Big|_{v^i = V^i(q^A, p_i, v^\alpha)}, \quad (15)$$

where

$$h_\alpha(q^A, p_i) = - \frac{\partial L(q^A, v^A)}{\partial v^\alpha} \Big|_{v^i = V^i(q^A, p_i, v^\alpha)}. \quad (16)$$

The functions $h_\alpha(q^A, p_i)$ are independent of the unresolved velocities v^α since $\text{rank } W_{AB} = r$. One should also take into account that now $\frac{dq^i}{dt} = V^i(q^A, p_i, v^\alpha)$ and $\frac{dq^\alpha}{dt} = v^\alpha$. Note that before imposing the Lagrange equations (15) the arguments of $L(q^A, v^A)$ were treated as independent variables.

A passage to Hamiltonian formalism can be done by the standard procedure: consider the full differential of both sides of (14) and use the supremum condition (12), which gives (till now the Lagrange equations of motion were not used)

$$\begin{aligned} \frac{\partial H_{mix}^{cl}}{\partial p_i} &= V^i(q^A, p_i, v^\alpha), \\ \frac{\partial H_{mix}^{cl}}{\partial p_\alpha} &= v^\alpha, \\ \frac{\partial H_{mix}^{cl}}{\partial q^i} &= - \frac{\partial L(q^A, v^A)}{\partial q^i} \Big|_{v^i = V^i(q^A, p_i, v^\alpha)} + [\bar{p}_\beta + h_\beta(q^A, p_i)] \frac{\partial v^\beta}{\partial q^i}, \\ \frac{\partial H_{mix}^{cl}}{\partial q^\alpha} &= - \frac{\partial L(q^A, v^A)}{\partial q^\alpha} \Big|_{v^i = V^i(q^A, p_i, v^\alpha)} + [\bar{p}_\beta + h_\beta(q^A, p_i)] \frac{\partial v^\beta}{\partial q^\alpha}. \end{aligned}$$

An application of the Lagrange equations (15) yields the system of equations which gives a Clairaut-Hamiltonian description of a singular theory

$$\frac{\partial H_{mix}^{cl}}{\partial p_i} = \frac{dq^i}{dt}, \quad (17)$$

$$\frac{\partial H_{mix}^{cl}}{\partial p_\alpha} = \frac{dq^\alpha}{dt}, \quad (18)$$

$$\frac{\partial H_{mix}^{cl}}{\partial q^i} = - \frac{dp_i}{dt} + [\bar{p}_\beta + h_\beta(q^A, p_i)] \frac{\partial v^\beta}{\partial q^i}, \quad (19)$$

$$\frac{\partial H_{mix}^{cl}}{\partial q^\alpha} = \frac{dh_\alpha(q^A, p_i)}{dt} + [\bar{p}_\beta + h_\beta(q^A, p_i)] \frac{\partial v^\beta}{\partial q^\alpha}. \quad (20)$$

This system has two disadvantages: 1) It contains the “nondynamical” momenta \bar{p}_α ; 2) It has derivatives of unresolved velocities v^α . To get rid of them, we introduce a “physical” Hamiltonian

$$H_0(q^A, p_i) = H_{mix}^{cl}(q^A, p_i, \bar{p}_\alpha, v^\alpha) - [\bar{p}_\beta + h_\beta(q^A, p_i)] v^\beta. \quad (21)$$

Using (12) and (14), one can show that the r.h.s. of (21) indeed does not depend on “nondynamical” momenta \bar{p}_α and unresolved velocities v^α . Then from (17)–(20) we obtain the system of (first order differential) equations which describes a singular Lagrangian theory

$$\frac{dq^i}{dt} = \{q^i, H_0(q^A, p_i)\} + \{q^i, h_\beta(q^A, p_i)\} v^\beta, \quad (22)$$

$$\frac{dp_i}{dt} = \{p_i, H_0(q^A, p_i)\} + \{p_i, h_\beta(q^A, p_i)\} v^\beta, \quad (23)$$

$$F_{\alpha\beta}(q^A, p_i) v^\beta = D_\alpha H_0(q^A, p_i), \quad (24)$$

where $\{X, Y\} = \frac{\partial X}{\partial q^i} \frac{\partial Y}{\partial p_i} - \frac{\partial Y}{\partial q^i} \frac{\partial X}{\partial p_i}$ is the “regular” Poisson bracket (in regular variables). We introduce a “ q^α -total derivative”

$$D_\alpha X = \frac{\partial X}{\partial q^\alpha} + \{X, h_\alpha(q^A, p_i)\} \quad (25)$$

and a “ q^α -non-Abelian field strength (curvature)”

$$F_{\alpha\beta}(q^A, p_i) = \frac{\partial h_\alpha(q^A, p_i)}{\partial q^\beta} - \frac{\partial h_\beta(q^A, p_i)}{\partial q^\alpha} + \{h_\alpha(q^A, p_i), h_\beta(q^A, p_i)\}. \quad (26)$$

The system (22)–(24) is equivalent to the Lagrange equations of motion due to our construction.

In the case $\text{rank } F_{\alpha\beta}(q^A, p_i) = n - r$, all the velocities v^α can be found from (24) in a purely algebraic way. If $\text{rank } F_{\alpha\beta}(q^A, p_i) = r_F < n - r$, then a singular theory has $n - r - r_F$ gauge degrees of freedom. In the first case one can resolve (24) as follows

$$v^\beta = D_\alpha H_0(q^A, p_i) \bar{F}^{\alpha\beta}(q^A, p_i), \quad (27)$$

where $\bar{F}^{\alpha\beta}(q^A, p_i)$ is the inverse matrix to $F_{\alpha\beta}(q^A, p_i)$, i.e. $F_{\alpha\beta}(q^A, p_i) \bar{F}^{\beta\gamma}(q^A, p_i) = \delta_\alpha^\gamma$. Substitute (27) to (22)–(23) to present the system of equations for a singular Lagrangian theory in the Hamiltonian form as follows

$$\frac{dq^i}{dt} = \{q^i, H_0(q^A, p_i)\}_F, \quad (28)$$

$$\frac{dp_i}{dt} = \{p_i, H_0(q^A, p_i)\}_F, \quad (29)$$

where we define a new bracket

$$\{X, Y\}_F = \{X, Y\} + \{X, h_\alpha(q^A, p_i)\} \bar{F}^{\alpha\beta}(q^A, p_i) D_\beta Y. \quad (30)$$

Note that the time evolution of any function X of dynamical variables (q^A, p_i) is also determined by the bracket (30) as follows

$$\frac{dX}{dt} = \{X, H_0(q^A, p_i)\}_F. \quad (31)$$

The bracket (30) is not anticommutative and does not satisfy Jacobi identity. Therefore, the standard quantization scheme is not applicable here. We expect that some more intricate further assumptions should be made to quantize consistently singular systems within the suggested approach.

To conclude, we describe Hamiltonian evolution of singular systems using $n - r + 1$ functions of dynamical variables $H_0(q^A, p_i)$ and $h_\alpha(q^A, p_i)$, a nonlinear Hamiltonian formalism. This is done by means of the generalized Clairaut-Legendre transform, that is by solving the corresponding multidimensional Clairaut equation. All variables are set as regular or degenerate. We consider the restricted phase space formed by the regular momenta p_i only.

There are two reasons why degenerate momenta \bar{p}_α are not worthwhile to be considered in a singular Lagrangian theory:

1) the mathematical reason: there is no possibility to resolve the degenerate velocities v^α as can be done for the regular velocities v^i in (12);

2) the physical reason: momentum is a “measure of movement”, but in degenerate directions there is no dynamics, hence — no reason to introduce the corresponding momenta at all.

Thus there is no notion of constraint [2, 10] as restriction on “nondynamical” momenta, because eventually we do not consider the latter — thus nothing to constrain. Under this approach, the degenerate coordinates q^α work as parameters analogous to $n - r$ time variables (with $n - r$ corresponding “Hamiltonians” $h_\alpha(q^A, p_i)$, see (25)). The Hamiltonian form of the equations of motion (28)–(29) is achieved by introducing a new bracket (30) depending on the above $n - r + 1$ functions. This bracket is responsible for the time evolution. However, is not anticommutative and does not satisfy Jacobi identity, and therefore its quantization requires non-Lie algebra methods.

Thus, we presented here a general construction. Further details and examples will appear elsewhere.

Acknowledgements. The author would like to express his deep thanks to V. P. Akulov, A. V. Antonyuk, Yu. A. Berezhnoj, V. Berezovoj, Yu. Bepalov, Yu. Bolotin, B. Broda, B. Burgstaller, B. Dragovic, V. Gershun, U. Günter, R. Jackiw, B. Jancewicz, H. Jones, A. T. Kotvytskiy, M. Krivoruchenko, G. C. Kurinnoj, M. Lapidus, J. Lukierski, P. Mahnke, N. Merenkov, B. V. Novikov, A. Nurmagambetov, L. A. Pastur, S. A. Ovsienko, M. Pavlov, S. V. Peletminskij, D. Polyakov, S. V. Prokushkin, S. Rauch-Wojciechowski, A. Razumny, V. Robuk, A. S. Sadovnikov, B. Shapiro, V. Shtelen, S. D. Sinel'shchikov, W. Siegel, K. S. Stelle, P. Urbanski, R. Wulkenhaar, A. A. Zheltukhin, M. Znojil and B. Zwiebach for fruitful discussions.

References

- [1] S. Duplij, "Analysis of constrained systems using the Clairaut equation," in *Proceedings of 5th Mathematical Physics Meeting: Summer School in Modern Mathematical Physics, 6 - 17 July 2008*, edited by B. Dragovich and Z. Rakic (Institute of Physics, Belgrade, 2009), p. 217 (arXiv:0804.2673).
- [2] P. A. M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University, New York, 1964).
- [3] J. F. Carinena, "Theory of singular Lagrangians," *Fortsch. Physik* **38**, 641 (1990).
- [4] W. M. Tulczyjew, "The Legendre transformation," *Ann. Inst. Henri Poincaré* **A27**, 101 (1977).
- [5] B. S. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon and Breach, London, 1965).
- [6] V. I. Arnold, *Mathematical methods of classical mechanics* (Springer, Berlin, 1989).
- [7] V. I. Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equations* (Springer-Verlag, New York, 1988).
- [8] W. M. Tulczyjew and P. Urbański, "A slow and careful Legendre transformation for singular Lagrangians," *Acta Phys. Pol.* **B30**, 2909 (1999).
- [9] H. Cendra, D. D. Holm, M. J. V. Hoyle, and J. E. Marsden, "The Maxwell-Vlasov equations in Euler-Poincaré form," *J. Math. Phys.* **39**, 3138 (1998).
- [10] K. Sundermeyer, *Constrained Dynamics* (Springer-Verlag, Berlin, 1982).